XXVII.

ON THE INFLUENCE OF INTERNAL FRICTION UPON THE CORRECTION OF THE LENGTH OF THE SECONDS' PENDULUM FOR THE FLEXIBILITY OF THE SUPPORT.

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IT has been shown by Professor A. M. Mayer that the only sensible resistance to the motion of a tuning-fork is proportional to the velocity. In the case of a slowly vibrating body, the chief effect is probably due to that lagging of the strain after the stress, which Weber has called the elastic after-effect (*Nachwirkung*). The influence of the former mode of resistance upon the period of oscillation of a pendulum oscillating on an elastic tripod is easily calculated. The same thing cannot, in my opinion, be effected for the other kind of resistance, in the present state of our knowledge; nevertheless, the main characteristics of the motion may be made out. Put

t, for the time;

φ, for the instantaneous angle of deflection of the pendulum;

s, for the instantaneous horizontal displacement of the knife-edge from its position of equilibrium, in consequence of the flexure of the support;

l, for the length of the corresponding simple pendulum;

h, for the distance from the knife-edge to the centre of mass of the pendulum;

g, for the acceleration of gravity;

γ, for the ratio of g to the statical displacement of the point of support, which would be produced by a horizontal force equal to the weight of the pendulum;

a, for the coefficient of internal friction supposed proportional to the velocity.

Then the differential equations are

$$lD^2_t\varphi + D^2_ts = -g\varphi$$
$$hD^2_t\varphi + D^2_ts = -\gamma s - aD_ts.$$

The solution of these equations will be of the form (using ϵ for the Neperian base and π for the ratio of circumference to diameter):

$$\left.\begin{aligned}\varphi &= A_1\epsilon^{z_1 t} + A_2\epsilon^{z_2 t} + A_3\epsilon^{z_3 t} + A_4\epsilon^{z_4 t},\\ s &= B_1\epsilon^{z_1 t} + B_2\epsilon^{z_2 t} + B_3\epsilon^{z_3 t} + B_4\epsilon^{z_4 t},\end{aligned}\right\} \qquad (1)$$

where z_1, z_2, z_3, z_4, are the roots of the equation

$$(l - h)z^4 + alz^3 + (\gamma l + g)z^2 + agz + \gamma g = 0,$$

where, for each subscript letter,

$$B = -\left(l + \frac{g}{z^2}\right) A,$$

and where four arbitrary constants are determined by the initial conditions.

The roots of the biquadratic equation are all imaginary, and may be written

$$z_1 = -\xi_1 + \eta_1\sqrt{-1} \qquad z_3 = -\xi_2 + \eta_2\sqrt{-1}$$
$$z_2 = -\xi_1 - \eta_1\sqrt{-1} \qquad z_4 = -\xi_2 - \eta_2\sqrt{-1}$$

Expressing the coefficients in terms of the real and imaginary parts of the roots, the equation becomes

$$z^4 + 2(\xi_1 + \xi_2)z^3 + (4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2)z^2$$
$$+ 2[(\xi_1^2 + \eta_1^2)\,\xi_2 + (\xi_2^2 + \eta_2^2)\xi_1]z + (\xi_1^2 + \eta_1^2)(\xi_2^2 + \eta_2^2) = 0.$$

If the terms in z^3 and z were neglected, that is, if a were neglected, the solution of the false equation so obtained would be as follows (where observe the varying sign of η_1): —

$$\text{False } z^2 = -\tfrac{1}{2}(4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2) \pm \tfrac{1}{2}(4\xi_1\xi_2 + \xi_1^2 + \xi_2^2$$
$$- \eta_1^2 + \eta_2^2)\sqrt{1 + 4\,\frac{4\xi_1\xi_2\eta_1^2 - \xi_1^2(\eta_2^2 - \eta_1^2) - \xi_1^2\xi_2^2}{(4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 - \eta_1^2 + \eta_2^2)^2}}.$$

Now, in the actual case, η_2 will be at least 100 times η_1, ξ_2 will be quite large, and ξ_1 very small. We may therefore neglect the square of the fraction under the radical; and we have very closely

$$\text{False } z_1^2 = \text{false } z_2^2 = -\eta_1^2 + \frac{4\xi_1\xi_2\eta_1^2 - \xi_1^2(\eta_2^2 - \eta_1^2) - \xi_1^2\xi_2^2}{4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 - \eta_1^2 + \eta_2^2}$$

$$\text{False } z_3^2 = \text{false } z_4^2 = -\eta_2^2 - \xi_1^2 - \xi_2^2 - 4\xi_1\xi_2 -$$
$$\frac{4\xi_1\xi_2\eta_1^2 - \xi_1^2(\eta_2^2 - \eta_1^2) - \xi_1^2\xi_2^2}{4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 - \eta_1^2 + \eta_2^2}$$

$$\text{False } z_1 = -\text{ false } z_2 = \eta_1\left(1 - \frac{1}{2\eta_1^2}\,\frac{4\xi_1\xi_2\eta_1^2 - \xi_1^2(\eta_2^2 - \eta_1^2) - \xi_1^2\xi_2^2}{4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 - \eta_1^2 + \eta_2^2}\right)\sqrt{-1}.$$

We thus see that, by neglecting the resistance, we get for the value of z_1 a quantity which requires only a minute correction in order to give the imaginary part of the true z_1. The same thing is not true for z_3 and z_4. Now, η_1 is ∂ divided by the principal period of oscillation of the pendulum upon the flexible stand. This is the quantity which we wish to determine; the others have only to be known approximately for the purpose of calculating the small correction to this. The logarithmic decrement of the amplitude of oscillation of the pendulum in the unit of time, so far as it is due to internal friction, is the quantity ξ_1. After these two quantities have been approximately ascertained, we may approximate to the quantity $(\xi_2^2 + \eta_2^2)$ by means of the equation

$$(\xi_1^2 + \eta_1^2)(\xi_2^2 + \eta_2^2) = \frac{\gamma g}{l - h}.$$

Then, by eliminating a between the two equations

$$2(\xi_1 + \xi_2) = \frac{al}{l - h},$$

$$2[(\xi_1^2 + \eta_1^2)\xi_2 + (\xi_2^2 + \eta_2^2)\xi_1] = \frac{ag}{l - h},$$

we obtain ξ_2, and consequently η_2. The values so obtained must satisfy the equation

$$4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2 = \frac{\gamma l + g}{l - h}.$$

Before proceeding to the consideration of the elastic after-effect, I propose to apply the equations thus obtained to the calculation of the correction of the seconds' pendulum for the flexure of the stand, supposing the internal friction to be proportional to the velocity.

For the pendulum used by me we have the approximate values: —

$l = 1.00$; h (heavy end up) $= 0.30$; h (heavy end down) $= 0.70$; g (New York) $= 0.993 \times \partial^2 = 9.89$; $\gamma = \frac{1}{0.0002125} = 4706$; $\eta_1 = 1.00$.

The accompanying table shows that $\xi_1 = 0.000008$. From this, we calculate that with heavy end up $\xi_2 = 0.08$, $\eta_2 = 257$; with heavy end down $\xi_2 = 0.17$, $\eta_2 = 392$. From this, it appears that the cor-

rection of η_1 is absolutely insensible, or, in other words, the effect of resistance (supposed proportional to the velocity) vanishes. That this is nearly, in fact, the case for my instrument is shown by the circumstance that the times of oscillation upon stands of different rigidities agree with the values calculated in leaving the internal friction out of account.

U. S. Coast Survey. Pendulum. Decrement of Arc due to internal friction of brass of tripod. Pendulum was swung on brass tripod in Paris, Geneva, and Kew. On a stand ten times as stiff in Hoboken. The times of decrement given are the SUM *of the times with the heavy end up and heavy end down.*

Half amplitude.	Time decrement on		Time shortened by internal friction.	Ratio of shortening.	Decrement in one second.	Decrement due to internal friction in one second.	Mean arc.	Natural logarithmic decrement due to internal friction.
	Flexible stand.	Stiff stand.						
100′								
	1073^s	1095^s	+ 22^s	.022	0′.0186	.00023	90′	.0000025
80								
	706	762	+ 56	.080	0 .0142	.00114	75	.0000152
70								
	1927	1969	+ 42	.020	0 .0104	.00037	60	.0000062
50								
	1377	1254	Reject.					
40							Mean	.000008

The last interval is probably affected by an error in the graduation of the scale used on one of the stands.

M. Plantamour proposes to determine the effect of the internal friction of the pendulum-stand upon the correction for flexure, by means of the difference between the statical and dynamical flexure. He has made numerous observations, which, according to his own interpretation of them, would show that, if a pendulum be supported in a certain inclined position until the stand has had time to take its position of equilibrium under this force, and then be let go, the ratio of the amplitude of oscillation of the stand to that of the pendulum is not the initial one, but is very different from that. If this were the case, the motion of the stand and pendulum could not be represented, even approximately, in the form (1), for by those equations the logarithmic decrement of the oscillation of the stand is the same as that of the pendulum. It is true that the two parts of the oscillation (nearly in the natural periods of the pendulum and of the stand) have different logarithmic decrements; and, as the ratio of their amplitudes is not the

same for the stand and for the pendulum, a certain change in the total relative amplitude might occur in this way, but only an excessively minute one, nothing like what M. Plantamour thinks he has observed. But it is so improbable that the motions of the stand and pendulum depart much from the forms (1) that it would be wrong to accept M. Plantamour's results, until they are confirmed by a purely optical observation free from any possible influence from the machinery attached to the stand. Such an observation has been made by me; and, though I admit it was rather rough, it is entirely opposed to M. Plantamour's conclusions. Should the latter be confirmed, they would totally nullify the attempt to correct for the effect of flexure, as they would show the inapplicability of the analysis which has been proposed for the solution of that problem, without affording us much hope of being able to replace it; and it would seem to be necessary in that case to reject all the work which has been done with the reversible pendulum.

If the pendulum were started in the manner proposed, and if for any cause the amplitudes of pendulum and stand were altered in different ratios, there would be a perpetual force at work tending to restore the old ratio, so long as the phases of the motion were the same in the pendulum and stand. But, if the phases differed, a part of this force would go to diminishing the amplitudes, and would act so strongly in this way that there would be a rapid decrement on account of this circumstance. Suppose, for instance, that in the differential equations we were to put instead of $D_t^2 s$, $D_t^2 s_1$, where s_1 is the value of s at a time later than t by a constant. The result of this would be (neglecting terms involving a) that instead of the square of the exponent of the Neperian base being the sum of two negative quantities, one of them very small compared with the other, the smaller of these quantities would be multiplied by an imaginary root of unity. This would have but little effect on the imaginary part of the exponent of base, which determines the period; but it would add a considerable real part, which would represent a corresponding decrement of arc.

It seems difficult to conceive of a force which should greatly change the relative amplitudes of oscillation of the pendulum and stand, without at the same time producing an enormous decrement of the amplitude of oscillation, such as certainly does not exist. It is for those who believe that the existence of such a force has been experimentally proved to show how great an effect it would have upon the period of oscillation. M. Plantamour supposes that the formula given by me in my paper, "De l'influence de la flexibilité du trépied sur l'oscillation

du pendule à reversion," would still apply to such a case; but I am unable to see upon what ground.

Meantime, in the present state of the question, it appears to me that we must appeal to direct experiment to determine the difference between the time of oscillation on a stiff and on a flexible stand. Such experiments were given by me in the paper above mentioned, and I have since greatly multiplied experiments on a stiff stand, with the general result there announced, namely that the difference is slightly greater than my theory supposes (owing, perhaps, to neglecting the energy of movement of the support), and not smaller, as M. Plantamour's views would require.